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BLASCHKE'S ROLLING BALL THEOREM AND THE TRUDINGER-WANG MONOTONE BENDING

ARAM L. KARAKHANYAN

ABSTRACT. We revisit the classical rolling ball theorem of Blaschke for convex surfaces with positive curvature and show that it is linked to another inclusion principle in the optimal mass transportation theory due to Trudinger and Wang. We also discuss an application to reflector antennae design problem.

1. INTRODUCTION

In this note we give two applications of an inclusion principle known as the rolling ball Theorem of Blaschke. Let \mathcal{M} and \mathcal{M}' be two hypersurfaces in \mathbb{R}^d . We say that \mathcal{M} and \mathcal{M}' are *internally tangent* at $x \in \mathcal{M}$ if they are tangent at x and have the same outward normal. Denote by $\Pi_x \mathcal{M}$ the second fundamental form of \mathcal{M} at x and let $n(x)$ be the outward unit normal at x . Then we have

Theorem 1.1. *Suppose \mathcal{M} and \mathcal{M}' are smooth convex surfaces with strictly positive scalar curvature such that $\Pi_x \mathcal{M} \geq \Pi_{x'} \mathcal{M}'$ for all $x \in \mathcal{M}, x' \in \mathcal{M}'$ such that $n(x) = n'(x')$. If \mathcal{M} and \mathcal{M}' are internally tangent at one point then \mathcal{M} is contained in the convex region bounded by \mathcal{M}' .*

W. Blaschke [1], pp. 114-117 proved Theorem 1.1 for closed curves in \mathbb{R}^2 . D. Koutroufotis [5] generalized Blaschke's theorem for complete curves in \mathbb{R}^2 and complete surfaces in \mathbb{R}^3 . Later J. Rauch [9], by using Blaschke's techniques, proved this result for compact surfaces in \mathbb{R}^d and J.A. Delgado [3] for complete surfaces. Finally J. N. Brooks and J. B. Strantzen generalized Blaschke's theorem for non-smooth convex sets showing that the local inclusion implies global inclusion [2].

Observe that if \mathcal{M} and \mathcal{M}' are internally tangent at x , then a necessary condition for \mathcal{M} to be inside \mathcal{M}' near x is

$$(1.1) \quad \Pi_x(v) \geq \Pi'_x(v) \quad \text{for all } v \in \mathbb{T}_x \mathcal{M} \cong \mathbb{T}_x \mathcal{M}'.$$

The tangent planes are parallel because \mathcal{M} and \mathcal{M}' are internally tangent at x . Therefore Theorem 1.1 says that if for all $x \in \mathcal{M}, x' \in \mathcal{M}', x \neq x'$ with coinciding normals $n'(x') = n(x)$ such that after translating \mathcal{M} by $x - x'$ we have that the translated surface $\tilde{\mathcal{M}}$ is

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locally inside \mathcal{M}' then \mathcal{M} is globally inside \mathcal{M}' . In other words, the local inclusion implies global inclusion or M rolls freely inside \mathcal{M}' .

Our aim is to apply Theorem 1.1 to optimal transportation theory and reflector antennae design problems. More specifically, for a smooth cost function $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ (subject to some standard conditions including the weak **A3** condition) and a pair of bounded smooth convex domains $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^d$ such that \mathcal{U} is c -convex with respect to \mathcal{V} (see Definition 2.1 below), we would like to take $\mathcal{M} = \mathcal{U}$ to be the reference domain and

$$\mathcal{M}' = \mathcal{N} := \{x \in \mathbb{R}^d \text{ s.t. } c(x, y_0) = c(x, y_1) + a\}, \quad y_1, y_2 \in \mathcal{V}$$

for some constant a . Then \mathcal{M}' is the boundary of sub-level set of the cost function c . We prove that if $\partial\mathcal{U}$ is locally inside \mathcal{N} in above sense then $\partial\mathcal{U}$ is globally inside \mathcal{N} provided that the sets \mathcal{N} are convex for all y_1, y_2, a . The precise result is formulated in Theorem 3.1 below and applications in Section 4.

A local inclusion principle for $\partial\mathcal{U}$ and \mathcal{N} is proved by Neil Trudinger and Xu-Jia Wang in [12], see the inequality (2.23) there. It is then used to show that under the **A3** condition a local support function is also global, see [12] page 411. The proof is based on a monotone bending argument that gives yet another geometric interpretation of the **A3** condition.

2. PRELIMINARIES

2.1. Optimal transportation. In order to formulate the result in the context of optimal transportation theory we need some standard definitions. Let $c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a cost function such that $c \in C^4(\mathbb{R}^d \times \mathbb{R}^d)$ and $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^d$.

Definition 2.1.

- Let $u : \mathcal{U} \rightarrow \mathbb{R}$ be a continuous function. A c -support function of u at $x_0 \in \mathcal{U}$ is $\varphi_{x_0} = c(x, y_0) + a_0, y_0 \in \mathbb{R}^d$ such that the following two conditions hold

$$\begin{aligned} u(x_0) &= \varphi_{x_0}(x_0), \\ u(x) &\geq \varphi_{x_0}(x), x \in \mathcal{U}. \end{aligned}$$

- If u has c -support at every $x_0 \in \mathcal{U}$ then we say that u is c -convex in \mathcal{U} .
- c -segment with respect to a point $y_0 \in \mathbb{R}^d$ is the set

$$\{x \in \mathbb{R}^d \text{ s.t. } c_y(x, y_0) = \text{line segment}\}.$$

One may take in the above definition $\{x \in \mathbb{R}^d \text{ s.t. } c_y(x, y_0) = tp_1 + (1-t)p_0\}$ with $t \in [0, 1]$ and p_0, p_1 being two points in \mathbb{R}^d .

- We say that \mathcal{U} is c -convex with respect to $\mathcal{V} \subset \mathbb{R}^d$ if the image of the set \mathcal{U} under the mapping $c_y(\cdot, y)$ denoted by $c_y(\mathcal{U}, y)$ is convex set for all $y \in \mathcal{V}$. Equivalently, \mathcal{U} is c -convex with respect to \mathcal{V} if for any pair of points $x_1, x_2 \in \mathcal{U}$ there is $y_0 \in \mathcal{V}$ such that there is a c -segment with respect to y_0 joining x_1 with x_2 and lying in \mathcal{U} .

Definition 2.2. Let u be a c -convex function then the sub-level set of u at $x_0 \in \mathcal{U}$ is

$$(2.1) \quad S_{h,u}(x_0) = \{x \in \mathbb{R}^d \text{ s.t. } u(x) < c(x, y_0) + [u(x_0) - c(x_0, y_0)] + h\}$$

for some constant h .

Equivalently, $S_{h,u}(x_0) = \{x \in \mathcal{U} \text{ s.t. } u(x) < \varphi_{x_0}(x) + h\}$ where φ_{x_0} is the c -support function of u at $x_0 \in \mathcal{U}$, see Definition 2.1.

Observe that in the previous definition one may take $u(x) = c(x, y_1)$ for some fixed $y_1 \neq y_0$.

Next we recall Kantorovich's formulation of optimal transport problem, see [13, 14]: Let $f : \mathcal{U} \rightarrow \mathbb{R}, g : \mathcal{V} \rightarrow \mathbb{R}$ be two nonnegative integrable functions satisfying the mass balance condition

$$\int_{\mathcal{U}} f(x) dx = \int_{\mathcal{V}} g(y) dy.$$

Then one wishes to minimize

$$(2.2) \quad \int_{\mathcal{U}} u(x) f(x) + \int_{\mathcal{V}} v(y) g(y) dy \rightarrow \min$$

among all pairs of functions $u : \mathcal{U} \rightarrow \mathbb{R}, v : \mathcal{V} \rightarrow \mathbb{R}$ such that $u(x) + v(y) \geq c(x, y)$.

It is well-known that a minimizing pair (u, v) exists [13, 14] and formally the potential u solves the equation

$$(2.3) \quad \det(u_{ij} - A_{ij}(x, Du)) = |\det c_{x_i, y_j}| \frac{f(x)}{(g \circ y)(x)}.$$

Here $A_{ij}(x, p) = c_{x_i x_j}(x, y(x, p))$ where $y(x, p)$ is determined from $D_x(c(x, y(x, p))) = p$.

Assume that c satisfies the following conditions:

- A1** For all $x, p \in \mathbb{R}^d$ there is unique $y = y(x, p) \in \mathbb{R}^d$ such that $\partial_x c(x, y) = p$ and for any $y, q \in \mathbb{R}^d$ there is unique $x = x(y, q)$ such that $\partial_y c(x, y) = q$.
- A2** For all $x, y \in \mathbb{R}^d$ $\det c_{x_i, y_j}(x, y) \neq 0$.
- A3** For $x, p \in \mathbb{R}^d$ there is a positive constant $c_0 > 0$ such that

$$(2.4) \quad A_{ij,kl}(x, p) \xi_i \xi_j \eta_k \eta_l \geq c_0 |\xi|^2 |\eta|^2 \quad \forall \xi, \eta \in \mathbb{R}^d, \xi \perp \eta.$$

A3 is the Ma-Trudinger-Wang condition [7].

J.Liu proved that if **A1-A3** hold then $S_{h,u}(x_0)$ is c -convex with respect to y_0 [6]. There are cost functions satisfying the weak **A3**

$$(2.5) \quad A_{ij,kl}(x, p) \xi_i \xi_j \eta_k \eta_l \geq 0 \quad \forall \xi, \eta \in \mathbb{R}^d, \xi \perp \eta.$$

i.e. when $c_0 = 0$ in (2.4), such that the corresponding sub-level sets are convex in classical sense, see Section 4.

We also remark that the condition **A3** is equivalent to

$$(2.6) \quad \frac{d^2}{dt^2} c_{ij}(x, y(x, p_t)) \xi_i \xi_j \geq c_0 |p_1 - p_0|^2$$

where x is fixed, $c_x(x, y(x, p_t)) = tp_1 + (1-t)p_0, t \in [0, 1]$ $c_x(x, y) = p_1, c_x(x, y_0) = p_0$ (this determines the so-called c^* -segment with respect to fixed x), see [12].

2.2. Shape operator. If \mathcal{M} is a surface with positive sectional curvature then by Sacksteder's theorem [10] \mathcal{M} is convex. For $x \in \mathcal{M}$, let $n(x)$ be the unit outward normal at x ($n(x)$ points outside of the convex body bounded by \mathcal{M}). The Gauss map $x \rightarrow n(x)$ is a diffeomorphism of \mathcal{M} onto \mathbb{S}^d [15], where \mathbb{S}^d is the unit sphere in \mathbb{R}^d . The inverse map n^{-1} gives a parametrization of \mathcal{M} by \mathbb{S}^d . If \mathcal{M}' is another smooth convex surface, and $w \in \mathbb{S}^d$, then $n^{-1}(w)$ and $(n')^{-1}(w)$ are the points on \mathcal{M} and \mathcal{M}' with equal outward normals.

Let $F : \Omega \rightarrow \mathbb{R}^m$ be a smooth map on a set $\Omega \subset \mathbb{R}^d$ and $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ then

$$\partial_v F(x) = \sum_{i=1}^d v_i \frac{\partial F(x)}{\partial x_i}$$

is the directional derivative operator.

We view the tangent space as a linear subspace of \mathbb{R}^d consisting of tangential directions. Then the tangent space $\mathbb{T}_x \mathcal{M}$ is the set of vectors perpendicular to $n(x)$.

Next we introduce the Weingarten map in order to define the second fundamental form. The Weingarten map $W_x : \mathbb{T}_x \rightarrow \mathbb{T}_x$ is defined by $W_x(v) = \partial_v n(x)$. \mathbb{T}_x is an inner product space (induced by the inner product in \mathbb{R}^d). Then W_x is self-adjoint operator on \mathbb{T}_x and the eigenvalues of W_x are the principal curvatures at x .

Remark 2.1. *Observe that since W_x is self-adjoint and \mathbb{T}_x is finite dimensional then there exists an orthonormal basis of \mathbb{T}_x consisting of eigenvectors of W_x .*

Definition 2.3. *The second fundamental form is defined as $\Pi_x(v, w) = W_x(v) \cdot w$. When $v = w$ we denote $\Pi_x(v)$.*

From definition it follows that if \mathcal{M} is parametrized by $r = r(u)$ and $x = r(u_0)$ then

$$(2.7) \quad \Pi_x(v) = -\partial_v^2 r \cdot n(x), \quad v \in \mathbb{T}_x$$

which readily follows from the differentiation of $n \cdot \partial_v r = 0$.

3. MAIN RESULT

Theorem 3.1. *Let $y_1, y_2 \in \overline{\mathcal{V}}$ and $\mathcal{N}(y_1, y_2, a) = \{x \in \mathbb{R}^d : c(x, y_0) = c(x, y_1) + a\}$ for some $a \in \mathbb{R}$ where c satisfies **A1**, **A2** and weak **A3**, see (2.5). Assume that \mathcal{N} is convex for all y_1, y_2, a and \mathcal{U} is convex domain with smooth boundary such that \mathcal{U} is c -convex with respect to \mathcal{V} , see Definition 2.1. If \mathcal{N} and $\partial \mathcal{U}$ are internally tangent at some point z_0 then \mathcal{U} is inside \mathcal{N} .*

Using the terminology of Blaschke's theorem it follows that under the conditions of Theorem 3.1 \mathcal{U} rolls freely inside \mathcal{N} . Observe that the c -convexity of sub-level sets is known under stronger condition **A3** [6]. In the next section we give an example of cost function c satisfying weaker form of **A3** (2.5) but such that \mathcal{N} is convex for all y_1, y_2, a . Proof to follow is inspired in [12].

Proof. Step 1: (Parametrizations)

To apply Theorem 1.1 we take $\mathcal{M} = \mathcal{U}$ and $\mathcal{M}' = \mathcal{N}$ and assume that \mathcal{U} and \mathcal{N} are internally tangent at z_0 . Assume that at $x'_0 \in \mathcal{N}$ and $x_0 \in \partial\mathcal{U}$ \mathcal{N} and $\partial\mathcal{U}$ have the same outward normal, see Figure 1.

In what follows we use the following radial parametrizations:

$$\begin{aligned} \partial\mathcal{U} & R(\zeta), \quad \zeta \in D_{\mathcal{U}}, \\ \mathcal{N} & \mathcal{X}(\omega), \quad \omega \in D_{\mathcal{N}}, \\ \partial(c_y(\partial\mathcal{U}, y_0)) & \rho(\zeta) = c_y(R(\zeta), y_0). \end{aligned}$$

Here $D_{\mathcal{U}}$ and $D_{\mathcal{N}}$ are the domains of corresponding parameters. Moreover, there are $\bar{\omega} \in D_{\mathcal{N}}$ and $\bar{\zeta} \in D_{\mathcal{U}}$ such that

$$(3.1) \quad x'_0 := \mathcal{X}(\bar{\omega}) \in \mathcal{N} \quad \text{and} \quad x_0 := R(\bar{\zeta}) \in \partial\mathcal{U}.$$

From now on $\bar{\zeta}$ and $\bar{\omega}$ are fixed. Let $\bar{n}(\bar{\zeta})$ denote the outward normal of the image $c_y(\mathcal{U}, y_0)$ at the point $\rho(\bar{\zeta})$. We have

$$(3.2) \quad \bar{n}^m(\bar{\zeta}) = c_{y_m, x_i}(R(\bar{\zeta}), y_0) n^i(\bar{\zeta}).$$

Observe that by assumption the constant matrix $\mu = [c_{y_m, x_i}(R(\bar{\zeta}), y_0)]^{-1}$ has non-trivial determinant, see **A2**. Furthermore, the set $\mu c_y(\mathcal{U}, y_0) = \{\mu x \text{ s.t. } x \in c_y(\mathcal{U}, y_0)\}$ is again convex because for any two points $q_1 = \mu z_1, q_2 = \mu z_2$ such that $q_1, q_2 \in \mu c_y(\mathcal{U}, y_0)$ and $z_1, z_2 \in c_y(\mathcal{U}, y_0)$ we have

$$\mu c_y(\mathcal{U}, y_0) \ni \mu(\theta z_1 + (1 - \theta)z_2) = \theta \mu z_1 + (1 - \theta) \mu z_2 = \theta q_1 + (1 - \theta)q_2$$

for all $\theta \in [0, 1]$.

Step 2: (Computing the second fundamental form of \mathcal{X})

Next, we introduce the vectorfield $r = r(\zeta), \zeta \in D_{\mathcal{U}}$ such that

$$(3.3) \quad r(\zeta) = \mu \rho(\zeta) = \mu c_y(R(\zeta), y_0).$$

We compute the first and second derivatives

$$(3.4) \quad r_{\zeta_s}^m := r_s^m = \mu_{\alpha\beta} c_{y_{\beta}, x_i} R_s^i,$$

$$(3.5) \quad r_{st}^m = \mu_{\alpha\beta} \left[c_{y_{\beta}, x_i x_j} R_s^i R_t^j + c_{y_{\beta}, x_i} R_{st}^i \right].$$

From (3.4) and (3.2) we see that at $r(\bar{\zeta})$ the normal is

$$(3.6) \quad n(\bar{\zeta}) = \mu \bar{n}(\bar{\zeta}).$$

Take $p_t = (1 - t)p_0 + tp_1, t \in [0, 1]$ and

$$(3.7) \quad p_t = c_x(x_0, y(x'_0, p_t)),$$

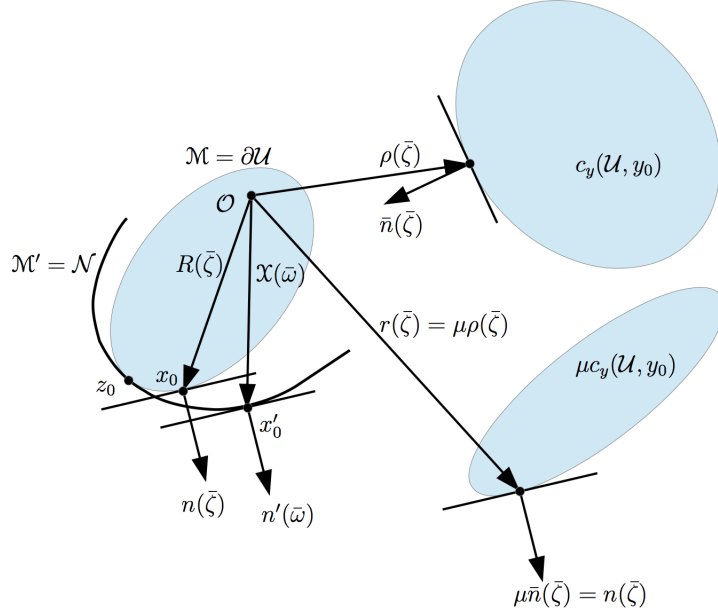


FIGURE 1. Schematic view to parametrizations of $\partial\mathcal{U}, \mathcal{N}, \partial(c_y(\mathcal{U}, y_0))$ and $\mu\partial(c_y(\mathcal{U}, y_0))$.

then $y_t := y(x'_0, p_t)$ defines the c -segment joining y_0 and y_1 , see **A2**. In particular, one has

$$\begin{aligned}
 (3.8) \quad p_1^i - p_0^i &= c_{x_i, y_m}(x_0, y(x'_0, p_t)) \frac{d}{dt} y^m(x'_0, p_t) \\
 &= \frac{d}{dt} y^m(x'_0, p_t) c_{y_m, x_i}(x_0, y(x'_0, p_t)).
 \end{aligned}$$

Let $\mathcal{X}^t(\omega)$ be the parametrization of $\mathcal{N}(t) = \{x \in \mathcal{U} : c(x, y_0) = c(x, y_t) + a\}$ (recall that $\mathcal{N}(t)$ is convex as the boundary of sub-level set). We can choose $a = a(t)$ so that all $\mathcal{N}(t)$ pass through the point x'_0 , in other words there is $\bar{\omega}^t$ such that $\mathcal{X}^t(\bar{\omega}^t) = x'_0$. Moreover, by (3.7) it follows that

$$\begin{aligned}
 (3.9) \quad c_{x_i}(\mathcal{X}^t(\bar{\omega}^t), y_0) - c_{x_i}(\mathcal{X}^t(\bar{\omega}^t), y_t) &= c_{x_i}(x'_0, y_0) - c_{x_i}(x'_0, y_t) \\
 &= p_0^i - p_t^i \\
 &= t(p_0^i - p_1^i).
 \end{aligned}$$

After fixing t and differentiating the identity $c(\mathcal{X}^t(\omega), y_0) = c(\mathcal{X}^t(\omega), y_t) + a(t)$ in ω we get

$$\begin{aligned}
 &[c_{x_i}(\mathcal{X}^t, y_0) - c_{x_i}(\mathcal{X}^t, y_t)] \mathcal{X}_{\omega_k}^{i,t} = 0, \\
 (3.10) \quad &[c_{x_i x_j}(\mathcal{X}^t, y_0) - c_{x_i x_j}(\mathcal{X}^t, y_t)] \mathcal{X}_{\omega_l}^{j,t} \mathcal{X}_{\omega_k}^{i,t} + [c_{x_i}(\mathcal{X}^t, y_0) - c_{x_i}(\mathcal{X}^t, y_t)] \mathcal{X}_{\omega_k \omega_l}^{i,t} = 0.
 \end{aligned}$$

Thus the normals of $\mathcal{N}(t)$ at x'_0 are collinear to $p_1 - p_0$ for all $t \in [0, 1]$, that is

$$(3.11) \quad n(x_0) = n'(x'_0) = \frac{p_1 - p_0}{|p_1 - p_0|}, \quad \mu \bar{n} = n \quad (\text{recall (3.6)}).$$

Hence we can rewrite (3.10) as follows

$$(3.12) \quad [(c_{x_i x_j}(\mathcal{X}^t, y_0) - c_{x_i x_j}(\mathcal{X}^t, y_t))] \mathcal{X}_{\omega_l}^{j,t} \mathcal{X}_{\omega_k}^{i,t} = -t(p_0^i - p_1^i) \mathcal{X}_{\omega_k \omega_l}^{i,t}.$$

Keeping $\mathcal{X}^t(\bar{\omega}^t) = x'_0$ fixed for all $t \in [0, 1]$, dividing both sides of the last identity by t and then sending $t \rightarrow 0$ we obtain

$$(3.13) \quad -[y'(x'_0, p_0) c_{y, x_i x_j}(x'_0, y_0)] \mathcal{X}_{\omega_l}^{j,t=0} \mathcal{X}_{\omega_k}^{i,t=0} = -(p_0^i - p_1^i) \mathcal{X}_{\omega_k \omega_l}^{i,t=0}.$$

On the other hand from (3.8) we see that $\frac{d}{dt} y(x'_0, p_t)|_{t=0} = (p_1 - p_0)\mu$. Thus substituting this into the last equality we obtain

$$(3.14) \quad \begin{aligned} [(p_1 - p_0)\mu c_{y, x_i x_j}(x'_0, y_0)] \mathcal{X}_{\omega_l}^{j,t=0} \mathcal{X}_{\omega_k}^{i,t=0} &= (p_0^i - p_1^i) \mathcal{X}_{\omega_k \omega_l}^{i,t=0} \\ &= -(p_1^i - p_0^i) \mathcal{X}_{\omega_k \omega_l}^{i,t=0} \end{aligned}$$

or equivalently

$$(3.15) \quad [n^\alpha \mu_{\alpha\beta} c_{y\beta, x_i x_j}(x'_0, y_0)] \mathcal{X}_{\omega_l}^{j,t=0} \mathcal{X}_{\omega_k}^{i,t=0} = -n^i \mathcal{X}_{\omega_k \omega_l}^{i,t=0}$$

if we utilize (3.11).

Step 3: (Monotone bending)

By Remark 2.1 we assume that $\mathbb{T}_{x_0} \partial \mathcal{U}$ and $\mathbb{T}_{x'_0} \mathcal{N}(t=0)$ have the same local coordinate system (by reparametrizing $\mathcal{N}(t=0)$ if necessary). From convexity of $\mu c_y(\mathcal{U}, y_0)$ boundary of which is parametrized by r we have

$$(3.16) \quad \begin{aligned} 0 &\geq r_{st}^\alpha n^\alpha = \mu \rho_{st} n = \mu_{\alpha\beta} (c_{y\beta, x_i x_j} R_s^i R_t^j + c_{y\beta, x_i} R_{st}^i) n^\alpha \\ &= \mu_{\alpha\beta} c_{y\beta, x_i x_j} R_s^i R_t^j n^\alpha + R_{st}^i n^i \\ &\stackrel{(3.15)}{=} -n^i \mathcal{X}_{\omega_k \omega_l}^{i,t=0} + R_{st}^i n^i. \end{aligned}$$

Now (2.6) yields that at x'_0

$$(3.17) \quad n^i \mathcal{X}_{\omega_k \omega_l}^{i,t} \geq n^i \mathcal{X}_{\omega_k \omega_l}^{i,t=0} \stackrel{(3.16)}{\geq} R_{st}^i n^i.$$

Recalling (2.7) we finally obtain the required inequality

$$\mathbb{I}_{x'_0} \mathcal{N} \leq \mathbb{I}_{x_0} \partial \mathcal{U}.$$

The proof is now complete. \square

Note that weak **A3** (i.e. when $c_0 = 0$ in (2.6)) is enough for the monotonicity to conclude the inequality $n^i \mathcal{X}_{\omega_k \omega_l}^{i,t} \geq n^i \mathcal{X}_{\omega_k \omega_l}^{i,t=0}$.

4. APPLICATIONS

4.1. Convex sub-level sets. There is a wide class of cost functions for which the set \mathcal{N} is convex. Observe that $c(x, y) = \frac{1}{p}|x - y|^p$ satisfies **A3** for $-2 < p < 1$ and weak **A3** if $p = \pm 2$ [7].

It is useful to note that if $\Omega_\psi = \{x \in \mathbb{R}^d \text{ s.t. } \psi(x) < 0\}$ for some smooth function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\Omega_\psi \neq \emptyset$ then

$$(4.1) \quad \partial^2 \psi(x) \tau(x) \cdot \tau(x) \geq 0, \quad \forall \tau(x) \in \mathbb{T}_x$$

is a necessary and sufficient condition for Ω_ψ to be convex provided that $\frac{\nabla \psi}{|\nabla \psi|}$ is directed towards positive ψ .

Using this, one can check that the boundaries of sub-level sets (e.g. $p = -2$)

$$\frac{1}{|x - y_2|^2} - \frac{1}{|x - y_1|^2} - a = 0$$

are convex. Here $\psi(x) := |x - y_2|^{-2} - |x - y_1|^{-2} - a < 0$ defines the sub-level set. Indeed, using (4.1) we see that for any unit vector τ perpendicular to the vector

$$\nabla \psi(x) = -\frac{2(x - y_2)}{|x - y_2|^4} + \frac{2(x - y_1)}{|x - y_1|^4}$$

we get

$$\begin{aligned} \partial_{\tau\tau}^2 \psi &= -\frac{2}{|x - y_2|^4} \left[1 - \frac{4((x - y_2) \cdot \tau)^2}{|x - y_2|^2} \right] + \frac{2}{|x - y_1|^4} \left[1 - \frac{4((x - y_1) \cdot \tau)^2}{|x - y_1|^2} \right] \\ \nabla \psi \cdot \tau = 0 &\quad \frac{2}{|x - y_1|^4} - \frac{2}{|x - y_2|^4} - \frac{8((x - y_1) \cdot \tau)^2}{|x - y_1|^6} + \frac{8((x - y_1) \cdot \tau)^2}{|x - y_2|^6} \frac{|x - y_2|^8}{|x - y_1|^8} \\ &= \frac{2}{|x - y_1|^4} - \frac{2}{|x - y_2|^4} + \frac{8((x - y_1) \cdot \tau)^2}{|x - y_1|^6} \left[\frac{|x - y_2|^2}{|x - y_1|^2} - 1 \right] \\ &= \frac{2}{|x - y_1|^4} - 2 \left(\frac{1}{|x - y_1|^2} + a \right)^2 + \frac{8((x - y_1) \cdot \tau)^2}{|x - y_1|^6} \left[\frac{\frac{1}{|x - y_1|^2}}{\frac{1}{|x - y_1|^2} + a} - 1 \right] \\ &= \frac{2}{|x - y_1|^4} - 2 \left(\frac{1}{|x - y_1|^2} + a \right)^2 - \frac{8((x - y_1) \cdot \tau)^2}{|x - y_1|^6} \frac{a}{\frac{1}{|x - y_1|^2} + a}. \end{aligned}$$

Altogether, we infer that

$$\partial_{\tau\tau}^2 \psi \leq 0 \quad \text{if } a \geq 0.$$

If $a < 0$ then we can swap y_1 and y_2 to conclude that $\partial_{\tau\tau}^2 \psi > 0$. Some examples of sub-level sets of inverse quadratic cost function are illustrated in Figure 2.

4.2. Antenna design problems. In parallel reflector problem [4] one deals with the paraboloids of revolution

$$(4.2) \quad P(x, \sigma, Z) = \frac{\sigma}{2} + Z^{n+1} - \frac{1}{2\sigma}|x - z|^2$$

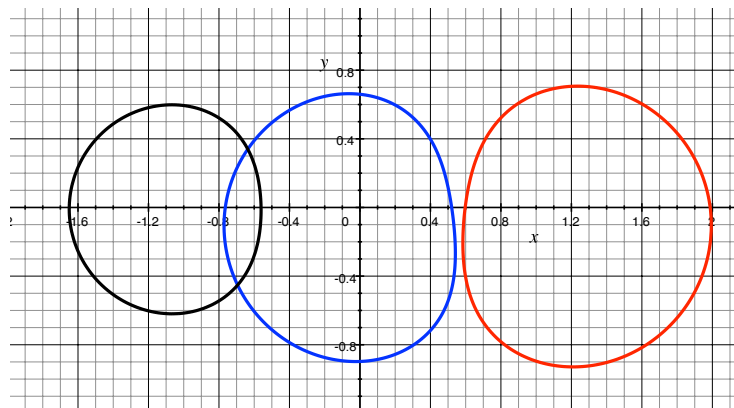


FIGURE 2. From left to right: $a = -2, y_1 = (-10^{-3}, 0), y_2 = (-1, -10^{-2})$; $a = 1, y_1 = (-10^{-1}, -10^{-1}), y_2 = (1, 10^{-2})$; $a = -1, y_1 = (-10^{-4}, 0), y_2 = (1.1, -10^{-1})$.

which play the role of support functions. Here the point $Z = (z, Z^{n+1}) \in \mathbb{R}^{n+1}$ is the focus of the paraboloid such that $\psi(z, Z^{n+1}) = 0$ for some smooth function ψ satisfying some structural conditions and σ is a constant. If P_1 is internally tangent to P_2 at z_0 and $\Pi_{z_0} P_1 \geq \Pi_{z_0} P_2$ then P_1 is inside P_2 , see Lemma 8.1 [4]. This again follows from Blaschke's theorem. Indeed, we have that at the points x and x' corresponding to coinciding outward normals

$$\Pi_x P_1 = \frac{1}{\sqrt{1 + |DP_1(x)|^2}} \frac{1}{\sigma_1} \delta_{ij}$$

and

$$\Pi_{x'} P_2 = \frac{1}{\sqrt{1 + |DP_2(x')|^2}} \frac{1}{\sigma_2} \delta_{ij}.$$

Furthermore $DP_1(x) = DP_2(x')$ and hence

$$(4.3) \quad \sqrt{1 + |DP_1(x)|^2} = \sqrt{1 + |DP_2(x')|^2}.$$

From $\Pi_{z_0} P_1 \geq \Pi_{z_0} P_2$ we infer that

$$(4.4) \quad \frac{1}{\sigma_1} \geq \frac{1}{\sigma_2}.$$

Consequently (4.4) and (4.3) imply that

$$\Pi_x P_1 \geq \Pi_{x'} P_2.$$

4.3. Another inclusion principle. There are various inclusion principles in geometry, we want to mention the following elementary one due to Nitsche [8]: Each continuous closed curve of length L in Euclidean 3-space is contained in a closed ball of radius $R < L/4$. Equality holds only for a "needle", i.e., a segment of length $L/2$ gone through twice, in opposite directions. Later J. Spruck generalized this result for compact Riemannian manifold \mathcal{M} of dimension $n \geq 3$ as follows: if the sectional curvatures $K(\sigma) \geq 1/c^2$ for

all tangent plane sections σ then \mathcal{M} is contained in a ball of radius $R < \frac{1}{2}\pi c$, and this bound is best possible. We remark here that there is a smooth surface $S \subset \mathbb{R}^3$ such that the mean curvature $H \geq 1$ and the Gauss curvature $K \geq 1$ then the unit ball cannot be fit inside S , see [11]. Notice that K is an intrinsic quantity and $H \geq 1$ implies that $K \geq 1$.

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